Derivation of tooth stiffness of asymmetric gears for loaded tooth contact analysis

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Abstract

Gears with asymmetric tooth forms can improve torque capacity of gearboxes if the load is largely unidirectional. For the asymmetric teeth, the primarily loaded flank can be optimized, typically by increasing the pressure angle to reduce contact stress, in order to improve the overall life. However, today's applications also have critical noise and efficiency requirements. With a higher pressure angle, power losses are normally reduced, but often at the cost of noise emission due to the resulting lower contact ratio. The transmission error, which can be evaluated with a loaded tooth contact analysis (LTCA), affects noise behavior of gears.

Modern LTCA software is using semi-analytical methods where tooth stiffness is calculated according to Weber-Banaschek (W/B) [1]. Therefore, in order to analyze the noise behavior of asymmetric gears without losing comparability to symmetric gears, it is necessary to extend the W/B [1] approach for asymmetric teeth as well.

In this paper, the main differences between the original tooth stiffness calculation according to W/B [1] and the approach of Langheinrich [2] for asymmetric teeth are shown. As a plausibility test root stress and stiffness of an almost symmetric tooth form is compared with a truly symmetric tooth form. Further the enhanced W/B formulas are compared with FEM results. Additionally, a special load case, load close to the root fillet, and its effects on the tooth stiffness calculation is examined.

1. Introduction

The use of asymmetric gears in automotive and other applications is a modern trend. However, only a few applications (produced in bigger series) are known to have asymmetric teeth. Still, an increased interest in asymmetric gears can be seen and companies have started to design and test such applications [5].

The pressure angle in the normal section of an asymmetric gear is different for left and right flank. This can be used as an advantage compared to symmetric gears since higher pressure angle increases the tooth flank (pitting) load capacity [6]. For gears loaded mostly in one direction, the loaded flank of the tooth can be designed with a high pressure angle and the unloaded flank, or seldomly loaded, with a low pressure angle. For a given gear pitch, using a low pressure angle on one flank permits the loaded flank to have a higher pressure angle compared to a symmetric gear due to the increased top land that results from the lower pressure angle side.

The main benefits of asymmetric gears over symmetric are higher load capacity and better efficiency, in some specific cases, also lower noise and vibration [4, 6]. The main drawback is a more complex manufacturing process (for steel gears) and related costs.

2. Calculation of Tooth Deformation of symmetric Teeth

The tooth deformation calculation according to W/B [1] in the normal section of a tooth is assuming three deformation components, namely bending, tilting and Hertzian flattening (see Figure 1 and 2). The bending and shear deformation can be calculated as a single-sided fixed elastic beam with an arbitrary symmetric (Figure 1) shape over the full facewidth *b* (Note: Symbols and units used in this paper corresponds to the original W/B paper [1]):

$$\frac{1}{2}P\delta_b = \frac{1}{2}\int_0^{y_p} \frac{M^2}{\frac{E}{1-\nu^2}\cdot\frac{b}{12}(2x)^3}dy + \frac{1}{2}\int_0^{y_p} \frac{1.2F_{bx}^2}{G\,b\,2x}dy + \frac{1}{2}\int_0^{y_p} \frac{F_{by}^2}{\frac{E}{1-\nu^2}b\,2x}dy \qquad (1)$$

The term $2x$, in Equation (1), and $\bar{S}$, in Equation (2), indicate that W/B [1] is assuming symmetric teeth with their middle line or line of symmetry on the y-axis. The deformation of the fixpoint (tilting) for steel gears is calculated according to:

$$\frac{1}{2}P\delta_t = \frac{9(1-\nu^2)}{\pi E b \bar{s}^2}M^2 + 2\frac{(1+\nu)(1-2\nu)}{2Eb\bar{s}}MF_{bx} + \frac{2.4(1-\nu^2)}{\pi E b}F_{bx}^2 + 0.294\tan\alpha_F\frac{2.4(1-\nu^2)}{\pi E b}F_{by}^2 \qquad (2)$$

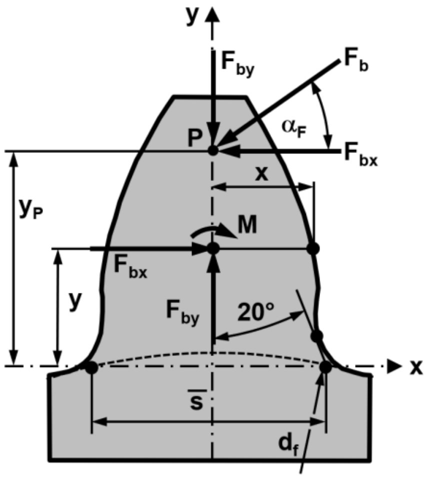

Figure 1: Model of a symmetric tooth to determine bending and tilting in the normal section [2]

The Hertzian flattening is calculated by representing the two mating gears as cylinders with the same facewidth, b_g. The cylinders radii are derived from the local curvature radii at the mating gears contact point with the assumed elongations t_1 and t_2 (Figure 2). These two terms, t_1 and t_2, are defined by the intersection point of the line of action and the respective tooth's middle line on the y-axis of the tooth.

$$\delta_{H1,2} = \frac{F_{bti}}{\pi b_g}\left[\left|\frac{1-\nu_1^2}{E_1}\ln\left(\frac{b_H^2}{4t_1^2}\right)+\frac{\nu_1(1+\nu_1)}{E_1}\right|+\left|\frac{1-\nu_2^2}{E_2}\ln\left(\frac{b_H^2}{4t_2^2}\right)+\frac{\nu_2(1+\nu_2)}{E_2}\right|\right] \quad (3)$$

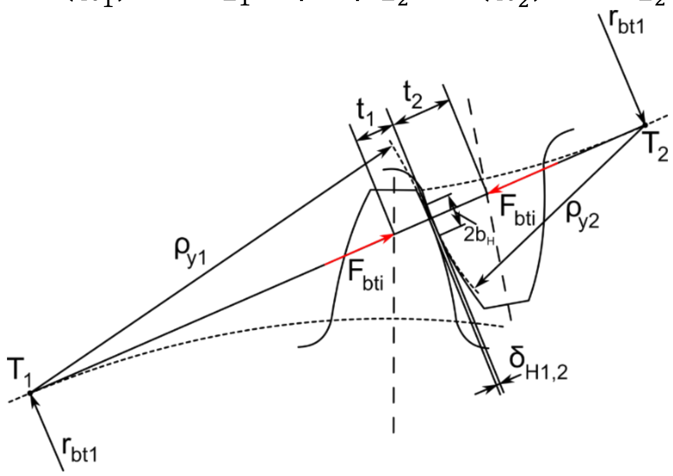

Figure 2: Model of a symmetric tooth pair to determine Hertzian flattening [5]

All deformations of a meshing tooth pair are calculated on the path of contact, so they can be summed up to the total deformation:

$$\delta = \delta_{b1} + \delta_{b2} + \delta_{t1} + \delta_{t2} + \delta_{H1,2} \quad (4)$$

3. Adapting the deformation calculation for asymmetric teeth

Due to the shape of an asymmetric tooth, it is necessary to consider several modifications to the fixpoint *M*, the chordal tooth thickness $\bar{S}_{Zy}$, and the force application points *P* and *V*, which affect the deformation of a tooth pair in contact.

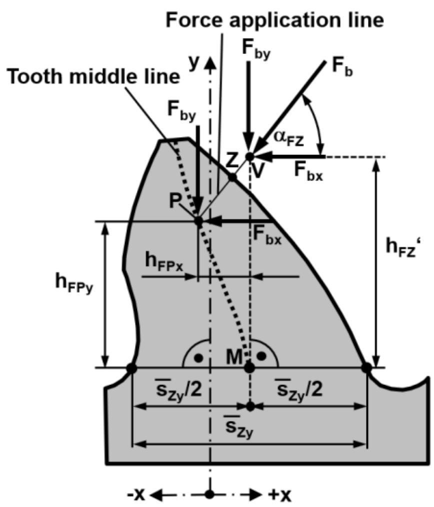

Figure 3: Model of an asymmetric tooth to determine bending and tilting [2]

The fixpoint *M*

W/B [1] defines the differentiation of the tooth from the gear body for symmetric teeth by the plane derived from the intersection point of the 20° tangent (to root fillet) with the root diameter d_f. The y-axis of the coordinate system is defined to be normal to the plane. The intersection point of the y-axis with the plane not only defines the half chordal tooth thickness $\bar{S}_{Zy}$, but also the fixpoint M (Figure 3). However, in the case of asymmetric teeth, this model of W/B [1] does not work. The intersection point of the 20° tangent has different tangent points with the root fillets of both flanks, and therefore different y-coordinates of the intersection point with the root diameter. Because of these different y-coordinates, the y-axis is not defined as normal to the plane and the half chordal tooth thickness $\bar{S}_{Zy}$ anymore. The y-axis intersection point with the plane cannot be the fixpoint M. To resolve this, a plane must be defined which has the same intersection point with the y-axis as the root diameter d_f.

While the x-coordinate of fixpoint *M* is always on top of the nonlinear tooth middle line, and therefore always at half chordal tooth thickness, the y-coordinate of the fixpoint *M* can be calculated as

$$y_M = \frac{d_f}{2} \quad (5)$$

The force application points P and V

The force application point P for asymmetric teeth is essentially the same as for symmetric teeth. The only difference is that point P is the crossing point of the line of action (which is also the elongation of the normal of the contact point Z at the tooth) with the nonlinear middle line instead of the y-axis (Figure 3). The force application point V represents a virtual contact point which has a perpendicular lever h_{FZ}' to the tooth/gear body differentiation plane (described in the previous paragraph) and has the same x-coordinate as the fixpoint M ($x_P = x_M$).

The resulting bending and tilting equations

The bending deformation component, according to W/B [1] can be adapted for asymmetric gears as follows:

$$\frac{1}{2}P\delta_b = \frac{1}{2}\int_0^{h_{FPy}} \frac{M^2}{\frac{E}{1-\nu^2}\cdot\frac{b}{12}\bar{s}_{Zy}{}^3}dy + \frac{1}{2}\int_0^{h_{FPy}} \frac{1.2F_{bx}^2}{Gb\bar{s}_{Zy}}dy + \frac{1}{2}\int_0^{h_{FPy}} \frac{F_{by}^2}{\frac{E}{1-\nu^2}b\bar{s}_{Zy}}dy \qquad (6)$$

The adapted single-sided fixed elastic beam equation accounts for the asymmetric tooth shape, and therefore asymmetric chordal tooth thickness, by transforming the term $2x$ from Equation (1) into $\bar{S}_{Zy}$ and considering the bending moment M as $M = F_{bx} \cdot h'_{FZ} = F_{bx}(y_V - y_M)$. With the given coordinates of contact point Z, the y_V can be calculated by $y_V = y_Z + (x_M - x_Z)\tan\alpha_{FZ}$.

The deformation of the fixpoint (tilting) can be done using Equation (2). The difference consists in the calculation of the bending moment $M = F_{bx}(y_V - y_M)$, with y_M according to Equation (5). Additional, Hertzian flattening can be calculated using Equation (3), except the assumed elongations t_1 and t_2 as defined by the intersection point P of the line of action with the nonlinear tooth middle line (Figure 4).

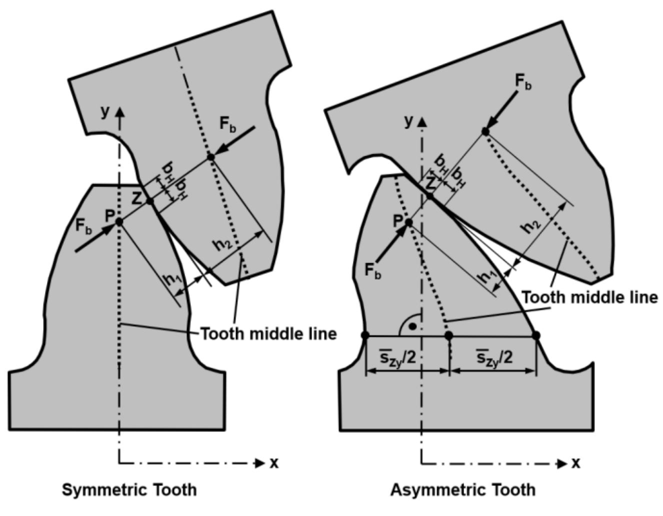

Figure 4: Model of Hertzian contact for symmetric and asymmetric tooth pair [2]

4. Comparison of asymmetric and symmetric gears

4.1. Principal effects of asymmetric teeth

In order to compare the stiffness calculated with the enhanced W/B approach, results for an asymmetric teeth with a slight difference in pressure angle (19.9999/20°, Table 1) is compared with a symmetric tooth (20°). The resulting tooth stiffness, however, is significantly affected (up to 10% difference), for this almost symmetric tooth. This is because the tooth bending stiffness according to W/B [1] considers the lever between the force application point *P* (or *V* for asymmetric teeth), and the fixpoint $M = F_{bx} \cdot h'_{FZ} = F_{bx}(y_V - y_M)$ as Table 2 documents. The different levers are caused by different methods used by W/B [1] and Langheinrich [2] to determine the fixpoint *M*. Figure 5 shows that the resulting bending stiffness and therefore the gear meshing stiffness is higher because of the difference in the force application lever calculation. Therefore the 19.9999°/20° case is stiffer than the symmetric tooth case. The bending stiffness calculated by FEM for some meshing positions is in the range between the symmetric and the asymmetric case. So, both methods are acceptable.

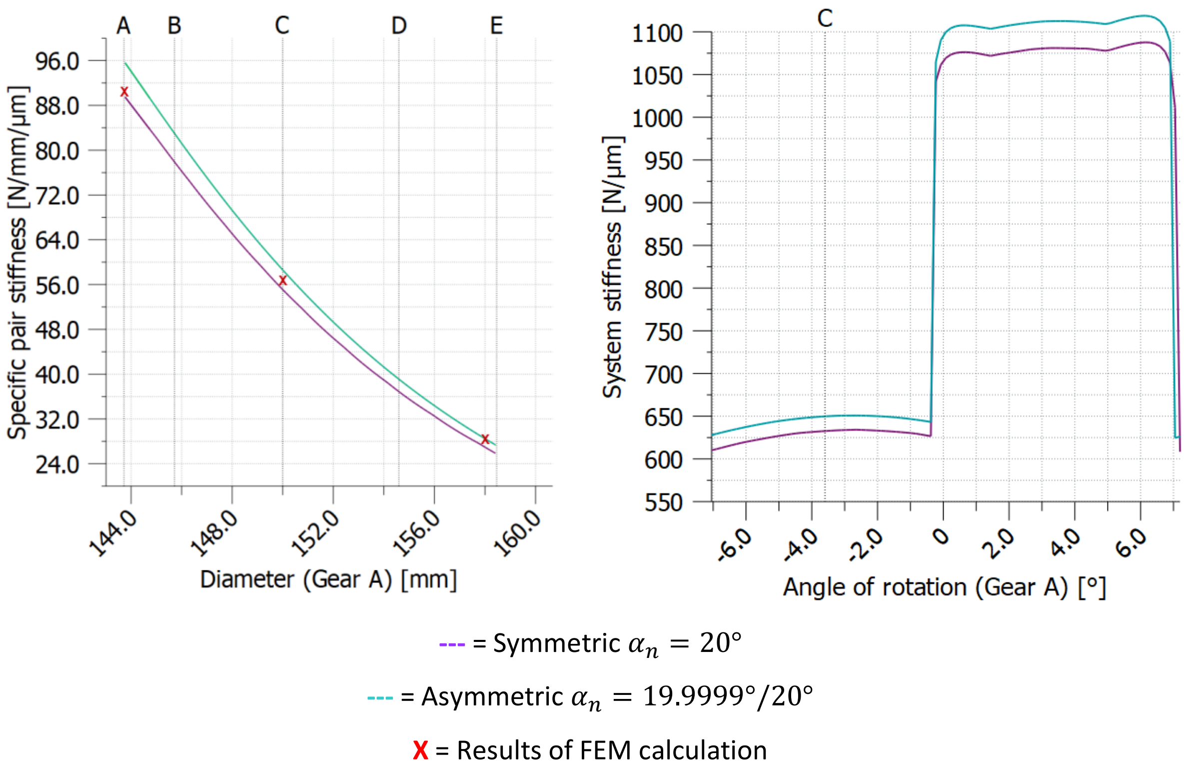

Figure 5: Comparison of symmetric and asymmetric single tooth tilting/bending stiffness and gear meshing stiffness

4.2. Implications on root stress

For the same examples, a LTCA is performed and the root stress is calculated. Root stress of the symmetric gear is calculated according ISO 6336-3 [2] (Figure 6a), for the asymmetric gear with the same method including Langheinrich's adaptions [2]. Because of the different derivations of the fixpoint *M*, and therefore the resulting difference of the force application lever h_{FZ}', the bending moment and resulting bending stiffness are different. However, the resulting root stress varies only slightly as shown in Table 2.

Table 1: Data of the spur gear example

α_{nL}/α_{nR} [°]	m_n [mm]	z_1/z_2	b [mm]	a [mm]	T_1 [Nm]	Flank in Contact
19.9999/20	6	25/76	44	303	1650	Right
h_{fP}^*	ρ_{fP}^*	h_{aP}^*				
0.955	0.38	0.705				

Table 2: Comparison of root stress, Hertzian pressure, force application lever, and fixpoint *M* at pitch point C, with F_n = 532 N/mm

Type	$y_P = h_{Fz}$ [mm]	$M(x/y)$ [mm]	$\sigma_{F\ (Y_SY_F)max}$ [N/mm²]	$\sigma_{F\ (Y_SY_F)30°}$ [N/mm²]	$(Y_SY_F)max$	$(Y_SY_F)30°$
Symmetric	4.68	0/68.80	201.23	188.23	2.4148	2.2588
Asymmetric	4.38	~0/69.11	201.22	188.22	2.4147	2.2586

The reason for the similar resulting root stresses is, that the root stress calculation according to ISO 6336 [3] does not consider the actual change of the lever between the force application point *P* and the fixpoint *M*. Instead, the ISO standard considers the lever (in the y-direction) between the force application point *P* and the point where a 30° tangent intersects the root fillet [3] as shown in Figure 6a. As the root stress calculated by ISO 6336 is accurate and well compared with FEM results [7], it is a comforting result, that the root stress of the gear calculated with the analytical method ISO 6336/Langheinrich is in perfect agreement with the symmetric case.

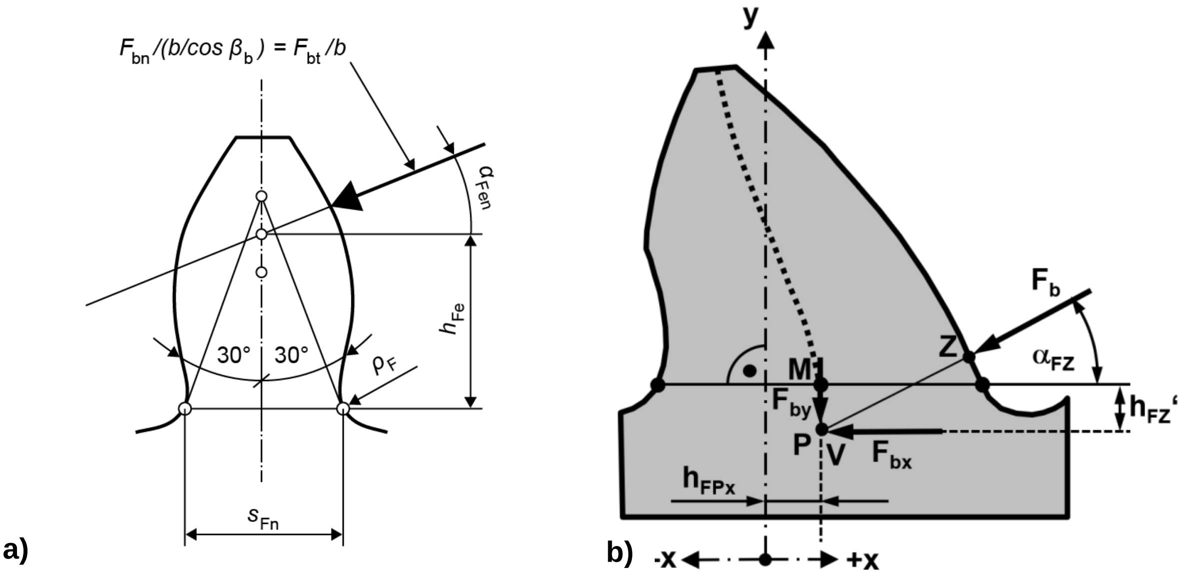

Figure 6: a) Determination of the normal chordal tooth thickness at the critical section [3], b) Model of an asymmetric tooth with a load applied close to the root fillet

4.3. Forces applied close to the root fillet

With asymmetric gears, an interesting load case is becoming more prominent, which can also occur for symmetric gears. As soon as the load application point Z approaches the root fillet, the load application lever h_{FZ}' becomes negative with respect to the fixpoint *M* (more prominent for gears with high pressure angles).

Table 3: Data of a spur gear with a load close to the root fillet

α_{nL}/α_{nR} [°]	m_n [mm]	z_1/z_2	b [mm]	a [mm]	T_1 [Nm]	Flank in Contact
10 /35	6	25/76	44	303	1650	Right
h^*_{fP1}	ρ^*_{fP1}	h^*_{aP1}	h^*_{fP2}	ρ^*_{fP2}	h^*_{aP2}	
0.955	0.1	0.705	0.955	0.38	0.9	

An FEM calculation of such a load case, Figure 6b Table 3, indicates that the middle line shifts slightly in the X- and Y-directions but apparently there is no bending and tilting between fixpoint *M* and force application point *P*. Figure 7 shows the respective FEM-Model used for the calculation and Figure 8 the normalized deformation of the tooth middle line.

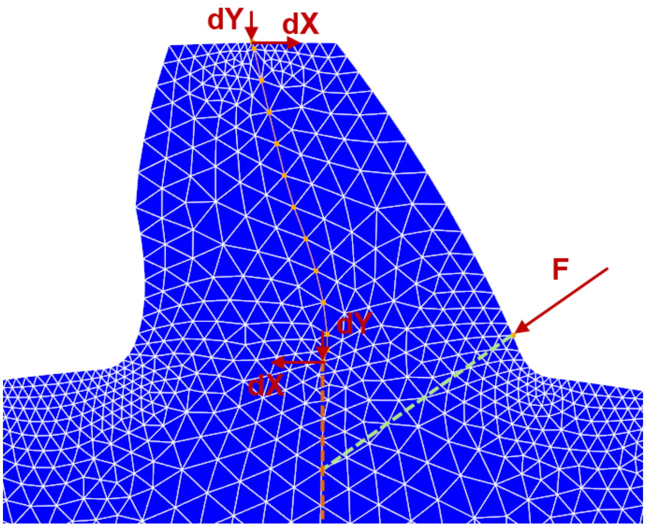

Figure 7: Annotated FEM-Model used for calculation of deformation

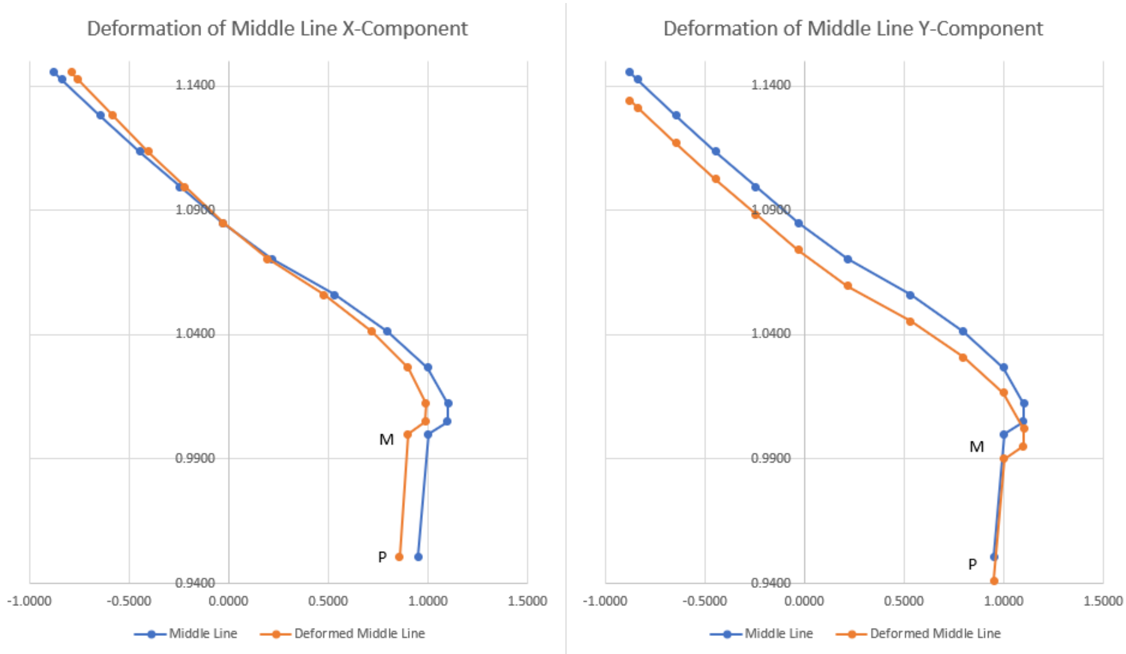

Figure 8: Results of FEM calculation

A special load case like a force applied close to the root fillet can, therefore, be calculated with a slightly simplified variant of W/B [1] Equations (1) and (2). While the bending results in $\frac{1}{2}P\delta_b = 0$ and tilting will be calculated with normal and shearing force only (9) (simplified for steel), the deflection due to Hertzian pressure can be considered as regular.

$$\frac{1}{2}P\delta_t = \frac{2.4(1-\nu^2)}{\pi Eb}F_{bx}^2 + 0.294 \tan\alpha_F \frac{2.4(1-\nu^2)}{\pi Eb}F_{by}^2 \quad (9)$$

5. Summary

Asymmetric gears have benefits with respect to flank safety for gears loaded mostly in one direction. As shown, asymmetric gears can be also considered in an LTCA. However, the equations of W/B [1] must be adapted by considering the real chordal tooth thickness $\bar{S}_{Zy}$ as shown in Equation (6). Results from Table 2 show, that the root stress calculation is only affected at the 30° tangent (according to [3]) and maximum $Y_S Y_F$ because of the difference in chordal tooth thickness $\bar{S}_{Zy}$, but not due to the difference of the force application lever h_{FZ} of asymmetric/symmetric gears as it might appear. However, the difference between the force application lever h_{FZ} affects the bending and tilting related tooth stiffness as well as the resulting gear meshing stiffness as shown in Figure 5. Therefore, calculations with W/B result in slightly higher tooth stiffnesses in the case of an almost symmetric gear. Asymmetric gears with a high pressure angle demonstrate a problem with the equations of W/B in case of a load close to the root fillet where the force application lever is negative with respect to the fixpoint *M* (Figure 6b). In such a case, only the tilting due to normal and shearing force should be considered as well as the Hertzian deflection. The bending of the asymmetric tooth due to the bending moment can be neglected as shown in Figure 8 and Equation (9).

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